Spectral Theory of Completely Positive Maps on C^*– and W^*-Algebras

The Single Operator Case
Draft Version

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Tübingen, 14.10.2010

Contents

Abstract The aim of this note is to study the spectral properties of completely positive maps on C*- and C*-algebras. These results are applied to convergence properties of the powers T^n. Since every positive operator on a commutative C*-algebra is completely positive (see e.g. Takesaki [25], IV. Cor. 3.5) the results in Schaefer [23], Chap. V. are obtained as a particular case.

1 Notation, Basic Results and a Review of the Perron-Frobenius Theory

1.1 Linear Operators and Spectrum

Let T be a continuous linear operator on a Banach space X . We denote by $\sigma(T)$ its *spectrum* and by $A\sigma(T)$ the *approximative point spectrum*, i.e. the set of all $\lambda \in \mathbb{C}$ such that there exists a sequence x_n in X with $\|x_n\| = 1$ and $\|(\lambda - T)x_n\| \to 0$. The approximative point spectrum contains the *point spectrum* $P\sigma(T)$. The *peripheral spectrum* of T is the set

$$\sigma_\pi(T) := \sigma(T) \cap \{\lambda \in \mathbb{C} \colon |\lambda| = r(T)\},$$

where $r(T)$ is the *spectral radius* of T. The *peripheral point spectrum* of T is the set

$$P\sigma(T) \cap \sigma_\pi(T).$$

If $\mu \in \rho(T)$, $\rho(T)$ the *resolvent set* of T, we denote by $R(\mu, T)$ the *resolvent* of T at μ. For $|\mu| > r(T)$ the resolvent is given by the *Neumann* series

$$R(\mu, T) = \sum_{k=0}^{\infty} \frac{1}{\mu^{k-1}} T^k.$$

1.2 C*– and W*-Algebras

The general reference to the theory of W*-algebras are the books of Takesaki [25] and Sakai [20].

By $\mathfrak{A}$ we shall denote a *C*-algebra* with unit $\mathbb{1}$, and we call $\mathfrak{A}$ a *W*-algebra*, if there exists a Banach space $\mathfrak{A}_*$ such that the dual space of $\mathfrak{A}_*$ is isomorphic to $\mathfrak{A}$. We call $\mathfrak{A}_*$ the *predual* of $\mathfrak{A}$, by $\mathfrak{A}^{sa}$ we denote the *self adjoint* part of $\mathfrak{A}$ and

$$\mathfrak{A}^+ := \{x^*x : x \in \mathfrak{A}\}$$

is its positive cone.

The set

$$\mathfrak{S} := \{\varphi \in \mathfrak{A}^* : \varphi \geqslant 0 \quad \text{and } \varphi(\mathbb{1}) = 1\}$$

is called the *state space* of $\mathfrak{A}$.

A face $\mathfrak{F}$ in $\mathfrak{A}^+$ is a subcone of $\mathfrak{A}^+$ such that

$$0 \leqslant x \leqslant y,\ y \in \mathfrak{F},\ x \in \mathfrak{A} \implies x \in \mathfrak{F}.$$

If $x \in \mathfrak{A}^+$, we denote by

$$\mathfrak{F}_x := \bigcup_{n \in \mathbb{N}} n\,[0,x], \quad [0,x] = \{y \in \mathfrak{A}^+ : 0 \leqslant y \leqslant x\},$$

the face generated by x. Note that the closure $\overline{\mathfrak{F}_x}$ is a face again.

1.3 Completely Positive Maps

An operattor $T \in \mathcal{L}(\mathfrak{A})$ is called *positive* (in symbols $T \geqslant 0$) if

$$T(\mathfrak{A}^+) \subseteq \mathfrak{A}^+.$$

It is called *n-positive* , if $T \otimes I_n$ is positive from the C*-algebra $\mathfrak{A} \otimes M_n$ to $\mathfrak{A} \otimes M_n$, where M_n is the C*-algebra of all complex $n \times n$-matrices.

Identifying $\mathfrak{A} \otimes M_n$ with the C*-algebra $M_n(\mathfrak{A}$ of all $n \times n$-matrices (x_{ij}) with $x_{ij} \in \mathfrak{A}$, this is equivalent to

$$T_n : (x_{ij}) \mapsto (T(x_{ij}))$$

beeing positive as operator on $M_n(\mathfrak{A})$.

If T is n-positive for all $n \in \mathbb{N}$, then we call T *completely positive.*

Every positive operator T satisfies

$$\|T\| = \|T(\mathbb{1})\| \quad \text{and} \quad T(a)^2 \leqslant \|T\| T(a^2)$$

for all $a \in \mathfrak{A}^{sa}$ (Kadison [15]).

If T is 2-positive, or even completely positive, then

$$T(x)T(x)^* \leqslant \|T\|T(xx^*) \tag{1.1}$$

for all $x \in \mathfrak{A}$ (Takesaki [25], III Cor. 3.8). Inequality 1.1 is called the *Schwarz inequality*. For a deeper discussion about this see e.g. the book of Paulsen [19].

We call a positive operator $T \in \mathcal{L}(\mathfrak{A})$ *irreducible* if no closed face of $\mathfrak{A}^+$, distinct from $\{0\}$ and $\mathfrak{A}^+$, is invariant under T.

Example 1.3.1. The identity map on a C*-algebra $\mathfrak{A}$ is irreducible, iff $\dim(\mathfrak{A}) = 1$.

Example 1.3.2. Let $\mathfrak{A} = M_n$, $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and u the unitary matrix in $\mathfrak{A}$ given by

$$u_{ij} = \begin{cases} \lambda^i & i = j, \\ 0 & i \neq j. \end{cases}$$

Then the operator $T := \mathrm{Ad}_u$ where $\mathrm{Ad}_u(x) = u^*xu$ is the inner *-automorphisms defined by u is not irreducible on $\mathfrak{A}$, since the projections p_i given by $p_i = e_i \otimes e_i$, e_i the canonical basis vectors of $\mathbb{C}^n$ are fixed vectors of T and hence the face generated by p_i is invariant under T.

Obviously, T is completely positive on $\mathfrak{A}$.

We call the triple $(\mathfrak{A}, \varphi, T)$ a *W*-dynamical system*, if $\mathfrak{A}$ is a W*-algebra, T is an identity preserving completely positive map on $\mathfrak{A}$ and φ is a faithful T invariant normal state on $\mathfrak{A}$. Since φ is faithful, T possesses a preadjoint T_* on $\mathfrak{A}_*$. The latter fact follows from Takesaki [25], III, Prop. 5.3.

1.4 The Classical Perron-Frobenius Theorem

Let $T = (a_{ij})$ be a matrix on $\mathbb{C}^n$ such that $a_{ij} \geqslant 0$ for all $1 \leqslant i, j \leqslant n$. If $r = r(T)$ and

$$r \cdot \alpha \in \sigma(T), |\alpha| = 1,$$

then

$$r \cdot \Gamma_\alpha \subseteq \sigma(T),$$

where Γ_α is the subgroup of the circle group generated by α. In particular, this implies that α is a root of unity unless $r = 0$ (see e.g. Schaefer [23], I, Thm. 2.7). More can be said if T is irreducible or *indecomposable*. Here, irreducible is equivalent to the fact that there is no permutation matrix P on $\mathbb{C}^n$ such that

$$PTP^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where T_{ii} are square matrices of order m_i $(i = 1, 2;\ 1 \leqslant m_i < n)$.

For a proof of the following result we refer to Schaefer [23], I, Thm. 6.5) or Wielandt [27].

Theorem 1.4.1 *Let $T \in \mathcal{L}(\mathfrak{A})$, $\mathfrak{A} = \mathbb{C}^n$ be positive and irreducible.*

a) The spectral radius r of T is greater then 0 and a simple eigenvalue for both T and the adjoint matrix ${}^tT$.

b) The peripheral spectrum of T is of the form $r \cdot \Gamma_{\mathrm{h}}$, where $\Gamma_{\mathring{h}}$ is the group of all h-roots of unity for some $\mathrm{h} \geqslant 1$.

c) Each $\alpha \in r \cdot \Gamma_{\mathrm{h}}$ is a simple root of the characteristic equation (hence a simple eigenvalue) of T.

d) The spectrum of T is invariant under the group of rotations (of the complex plane) corresponding to Γ_{h}.

There are applications of this theorem in many different branches of mathematics, e.g. graph theory, probability theory, numerical analysis etc.. For this see e.g. Bermann & Plemmons [2] or Schaefer [23], Chapter I.

Therefore it was only natural to ask how this theory can be generalized. In Schaefer [23], Chapter V, this has been done for Banach lattices, hence in particular for *commutative* C*-algebras. For a rich source of applications see Engel & Nagel [6] or Nagel [17].

1.5 Basic Spectral Properties of Positive Operators on C*-algebras

Since the positive cone of a C*-algebra is generating and has $\mathbb{1}$ as an interior point, the general theory for positive operators on such type of ordered

Banach spaces can be applied. We refer to Schaefer [22], Appendix.

Theorem 1.5.1 *Let $\mathfrak{A}$ be a C^*-algebra and $T \in \mathcal{L}(\mathfrak{A})$ a positive operator.*

a) The spectral radius $r(T)$ is an element of the spectrum of T.

b) If $r(T)$ is a pole of the resolvent, then it is of maximal order on the peripheral spectrum.

c) There exists a state φ on $\mathfrak{A}$ such that $T^*\varphi = r(T)\varphi$.

d) If T is compact with $r(T) > 0$, then there exists $x \in \mathfrak{A}_+$ with $Tx = r(T)x$.

Let us close this section with some examples.

Example 1.5.2. The Volterra operator on $C([0,1]$ shows that the condition $r(T) > 0$ can not be dropped in (d).

Example 1.5.3. The right shift on ℓ^∞ shows that there are positive operators with empty peripheral point spectrum on (commutative) C*-algebras.

Example 1.5.4. Let $\mathfrak{A}$ and T be as in Example 1.3.2, then $\sigma(T) = \{1, \lambda, \lambda^*\}$. This shows in contrast to the commutative situation, that the peripheral point spectrum is in general not cyclic. Recall that T is not irreducible.

Example 1.5.5. We consider the C*-algebra $\mathfrak{A} = M_2(\mathbb{C})$ and let $T \in \mathcal{L}(\mathfrak{A})$ be the mapping

$$x \mapsto \mathrm{Tr}(x)\mathbb{1} - x, \quad x \in \mathfrak{A}, \tag{1.2}$$

where Tr is the normalized trace on $\mathfrak{A}$. Then its characteristic polynomial is

$$\Delta(\lambda) = (\lambda + 1)^3(\lambda - 1)$$

and its minimal polynomial is

$$\mathrm{m}(\lambda) = (\lambda + 1)(\lambda - 1)$$

Then T is irreducible, but in contrast to the commutative situation, -1 is not a simple root of the characteristic polynomial.

Example 1.5.6. We consider again the C*-algebra $\mathfrak{A} = M_2(\mathbb{C})$ and let $T \in \mathcal{L}(\mathfrak{A})$ be defined as

$$T: \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \mapsto \begin{pmatrix} \alpha_{22} & \frac{1}{2}\alpha_{12} \\ \frac{1}{2}\alpha_{21} & \alpha_{11} \end{pmatrix} \tag{1.3}$$

Then T is positive and irreducible with spectrum $\sigma(T) = \{1, -1, 1/2\}$, i.e. the spectrum of T is not invariant under the rotation of the complex plan of $180°$.

The examples 1.5.5 and 1.5.6 are positive operators, which are not n-positive, $n \geqslant 2$ and example 1.5.4 is an inner automorphism which is completely positive but not irreducible.

2 Spectral Theory on C*-Algebras

2.1 The Peripheral Spectrum on Finite-Dimensional C*-algebra

If T is a positive matrix, then the peripheral spectrum is *cyclic* as mentioned above. As we have seen, this cannot be extended, even if we consider irreducible operators on a finite-dimensional C*-algebra. In this section we will investigate these phenomena more closely to understand the reasons for this behavior.

To carry out our analysis we need some facts from the theory of *Jordan algebras* and we refer for details to Braun & Köcher [3] and, more related to the theory of C*-algebras, to the book Alfsen & Schultz [1].

Since we are dealing with C*-algebra, we can restrict our attention to the class of formally real finite-dimensional Jordan algebras where by definition, a formally real finite–dimensional Jordan algebra $(\mathfrak{J}, \circ)$ is a finite-dimensional, commutative but not necessarily associative algebra with identity over the real numbers, such that

$$\begin{aligned} x \circ y &= y \circ x \\ x \circ (y \circ x^2) &= (x \circ y) \circ x^2, \\ \sum_{k=1}^{n} x_k^2 = 0 &\implies x_k = 0 \quad (1 \leqslant k \leqslant n) \end{aligned} \tag{2.1}$$

for all $x, y \in \mathfrak{J}, n \in \mathbb{N}$. Here $x^2 := x \circ x$,

We call $\mathfrak{J}$ a JB-algebra if there exists a norm on $\mathfrak{J}$ satisfying

1. $\|x \circ y\| \leqslant \|x\|\|y\|$,
2. $\|x^2\| = \|x\|^2$,
3. $\|x^2\| \leqslant \|x^2 + y^2$

for all $x, y \in \mathfrak{J}$.

Example 2.1.1. If $\mathfrak{A}$ is a C*-algebra, then the self adjoint part $\mathfrak{A}^{sa}$ of $\mathfrak{A}$ is a real JB-algebra with the product

$$a \circ b := \frac{1}{2}(ab + ba), \quad a, b \in \mathfrak{A}$$

and with the induced C*-norm .

Example 2.1.2. (Spin factor) As a more sophisticated example we have to introduce the so called *abstract spin factor*, introduced by (Topping [26]) and discussed in detail in Hanche-Olsen & Størmer [13], Chapter 6.

We start from a real Hilbert space $\mathfrak{H}$ with scalar product $(\cdot|\cdot)$, norm $\| \ \|_2$ and choose an arbitrary $v \in \mathfrak{H}$ with $\|v\|_2 = 1$. Then

$$\mathfrak{H} = [v] \oplus \{v\}^{\perp}.$$

Next we define a product $\circ$ on $\mathfrak{H}$ by

$$(\alpha v + x) \circ (\beta v + y) = (\alpha\beta + (x|y))v + (\alpha y + \beta v), \tag{2.2}$$

where $\alpha, \beta \in \mathbb{R}$ and $x, y \in \{v\}^{\perp}$. Then $(\mathfrak{H}, \circ)$ is a Jordan algebra and a JB-algebra with respect to the norm

$$\|\alpha v + x\| := |\alpha| + \|x\|_2. \tag{2.3}$$

The set

$$\mathfrak{H}_+ = \{\alpha v + x \colon \|x\|_2 \leqslant |\} \tag{2.4}$$

defines a positive cone in $\mathfrak{H}$ which is generating. This cone is also called the *ice cream cone* for obvious reasons.

If we take $U \in \mathcal{B}(\{\mathfrak{v}\}^{\perp})$ with $\|U\| \leqslant 1$ and define T on $\mathfrak{H}$ by

$$T := I_{|[v]} \oplus U,$$

we have T positive, $r(T) = 1$ and $\sigma(T) = \{1\} \cup \sigma(U)$. Thus $\sigma(T)$ can be any subset of the unit disk in $\mathbb{C}$ provided 1 belongs to it.

Example 2.1.3. Let $\mathfrak{A} = M_2(\mathbb{C})$ the C*-algebra of 2×2 matrices. By the isomorphism

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} + i\alpha_{21} \\ \alpha_{12} - i\alpha_{21} & \alpha_{22} \end{pmatrix} \mapsto \frac{1}{2}(\alpha_{11} + \alpha_{22}) + \left(\frac{1}{2}(\alpha_{21} - \alpha_{12}), \alpha_{12}, \alpha_{21}\right)$$

$\alpha_{ij} \in \mathbb{R}$, the the selfadjoint part of $\mathfrak{A}$ is isomorphic to the spin factor

$$\mathfrak{H} = [\mathbb{1}] \oplus \mathbb{R}^3.$$

On $\mathbb{R}^3$ we consider an orthogonal operator U with $\sigma(U) = \{-1, \lambda, \lambda^*\}$ with $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and define $T \in \mathcal{L}(\mathfrak{H})$ as

$$T = I + U.$$

Via the above isomorphism and because $\mathfrak{A} = \mathfrak{A}^{\mathrm{sa}} + i\mathfrak{A}^{\mathrm{sa}}$ we obtain an operator on $\mathfrak{A}$ which is positive, irreducible and has $\{-1, \lambda, \lambda^*\}$ in its spectrum.

This example indicates that the highly non-symmetrical behavior of the spectrum has to do with the availability of the spin-factor. But note that the example above and the inner automorphism in example 1.5.4 are different operators on $\mathfrak{A}$.

We now show under what conditions the Perron-Frobenius theorem can be extended to the non-commutative setting. Recall that a JB-algebra with only trivial ideals is called *simple*.

Lemma 2.1.4 *Let $\mathfrak{A}$ be a finite-dimensional C*-algebra, let $0 \leqslant T \in \mathcal{L}(\mathfrak{A})$ be irreducible. Then $r = r(T) > 0$ and $r^{-1}T$ is similar to a positive irreducible $S \in \mathcal{L}(\mathfrak{A})$ with $S\mathbb{1} = \mathbb{1}$.*

Theorem 2.1.5 *Let $\mathfrak{A}$ be a finite-dimensional C*-algebra, let $0 \leqslant T \in \mathcal{L}(\mathfrak{A})$ be irreducible with $T\mathbb{1} = \mathbb{1}$ and let $\mathfrak{M}$ be the linear span of all eigenvectors of T pertaining to the peripheral eigenvalues.*

1. *$\mathfrak{M}^{\mathrm{sa}} := \mathfrak{M} \cap \mathfrak{A}^{\mathrm{sa}}$ is a JB-subalgebra of $\mathfrak{A}^{\mathrm{sa}}$ and $T_0 := T_{|\mathfrak{M}^{\mathrm{sa}}}$ is a Jordan automorphism of $\mathfrak{M}^{\mathrm{sa}}$.*

2. *There exists (up to permutation) uniquely determined simple Jordan ideals* $\mathfrak{J}_0, \dots, \mathfrak{J}_{k-1}$ *in* $\mathfrak{M}^{\mathrm{sa}}$ *such that*

   a) $\mathfrak{M}^{\mathrm{sa}} = \bigoplus_{i=0}^{k-1} \mathfrak{J}_i$,

   b) $T_0(\mathfrak{J}_{\mathrm{i}}) = \mathfrak{J}_{\mathrm{i}+1},\ 0 \leqslant i \leqslant k,\ \mathfrak{J}_0 = \mathfrak{J}_{\mathrm{k}}$,

   c) $\mathfrak{J}_0$ *is either isomorphic to* $\mathbb{R}$ *or to the spin factor* $\mathbb{R} \oplus \mathbb{R}^s$ *for some* $s \geqslant 2$.

PROOF. The proof proceeds by the following steps.

1. We first show that for all $x \in \mathfrak{M}$ and $y \in \mathfrak{A}$ we have

$$T(x \circ y) = x \circ T(y) \quad \text{and} \quad T(y \circ x) = T(y) \circ x. \tag{2.5}$$

   This implies in particular that the linear span of all eigenvectors pertaining to the peripheral eigenvalues is closed with respect to the Jordan product and therefore $\mathfrak{M}^{\mathrm{sa}}$ is a real JB algebra and T_0 a Jordan automorphism.

2. By the theory of finite-dimensional Jordan algebras (see the original paper by Jordan, von Neumann & Wigner [14] or Hanche-Olsen & Størmer [13]), Lemma 2.9.4.), there exists a maximal k and uniquely determined (up to a permutation) simple Jordan Ideals $\mathfrak{J}_i$ such that $\mathfrak{M}^{\mathrm{sa}} = \bigoplus_{i=0}^{k-1} \mathfrak{J}_i$.

3. Since T_0 is still irreducible, we can assume $T_0(\mathfrak{J}_i) \subseteq \mathfrak{J}_{i+1}$ by $i \in \mathbb{Z}/(k)$.

4. Since $\mathfrak{M}^{\mathrm{sa}}$ is a finite-dimensional JB algebra, the results of Jordan, von Neumann & Wigner [14] can be applied (see the original paper by Jordan, von Neumann & Wigner [14] or Hanche-Olsen & Størmer [13]), Thm. 2.9.6. and Thm. 2.9.8). Using this classification and the result of Dieudonné [4] we obtain the proposition.

The details can be found in Groh [7]. ∎

Remark 2.1.6. We call the number k appearing in Proposition 2.1.5 the *index of imprimitivity* of T. Note that k is not the number of peripheral eigenvalues of T (see example 2.1.3)

Theorem 2.1.7 *let* $0 \leqslant T \in \mathcal{L}(\mathfrak{A})$ *be irreducible with index of imprimitivity k and spectral radius r.*

1. *There exists peripheral eigenvalues* $\alpha_1, \dots, \alpha_m$ *with* $\alpha_i \notin r \cdot \Gamma_k$, $1 \leqslant i \leqslant m$, *such that*

$$\sigma(T) \cap r \cdot \Gamma = \left(\bigcup_{i=1}^{m} \alpha_i \Gamma_k \right) \cup r \cdot \Gamma_k.$$

2. *Each* $\alpha \in r \cdot \Gamma_k$ *is a simple root of the characteristic polynomial of* T *and hence a simple eigenvalue.*

The proof of this theorem follows from Prop. 2.1.5 and the fact that the spectral radius of T is a simple eigenvalue. For details see Groh [7].

Remark 2.1.8. Prop. 2.1.5 shows that the asymmetric behavior of the spectrum is caused by the spin factor. We will therefore restrict our attention to the class of *completly positive* operators. In this case the spin factor will no longer appear.

2.2 The Peripheral Point Spectrum of Completely Positive Operators

As we have already mentioned in 1.5 some basic spectral properties hold for positive operators on C*-algebras. The most important point is that always $r(T) \in \sigma(T)$. From this the following follows.

Proposition 2.2.1 *Let* T *be a positive operator with spectral radius* $r(T)$ *on a* C*-*algebra* $\mathfrak{A}$. *Then* $r(T)$ *is an eigenvalue of the adjoint operator* T^* *with an eigenvector* φ *belonging to the state space* $\mathfrak{S}(\mathfrak{A})$ *of* $\mathfrak{A}$.

PROOF. Since $r(T) \in \sigma(T)$, there exists a positive linear form ψ on $\mathfrak{A}$, such that

$$\|R(\mu, T)^* \psi\| \nearrow \infty \quad \text{as} \quad \mu \searrow r(T).$$

Now choose an ultrafilter $\mathfrak{U}$ on $(r(T), \infty)$ converging to $r(T)$ and for $\mu > r(T)$ define

$$\psi_\mu := \lim_{\mathfrak{U}} \|R(\mu, T)^* \psi\|^{-1} R(\mu, T)^* \psi$$

Then

$$\varphi := \lim_{\mathfrak{U}} \psi_\mu$$

exists because of the weak*-compactness of the dual unit ball. Then $T^*\varphi = r(T)\varphi$ and $\varphi(\mathbb{1}) = 1$ because $\psi_\mu \in \mathfrak{S}(\mathfrak{A})$. ■

Remark 2.2.2. Let T be positive and irreducible on $\mathfrak{A}$ with spectral radius $r = r(T)$ and let φ be a state on $\mathfrak{A}$ with $T^*\varphi = r\varphi$.

1. One has necessarily $r(T) > 0$.

   The face $\mathfrak{F}_\varphi = \{0 \leqslant x \in \mathfrak{A} \colon \varphi(x) = 0\}$ is T-invariant and hence dense in $\mathfrak{A}_+$ or $== \{0\}$. But φ is a state which implies $\mathfrak{F}_\varphi = \{0\}$. But then $r\varphi(\mathbb{1}) = \varphi(T\mathbb{1}) = 0$ implies $T(\mathbb{1}) = 0$ or, because of $\|T\| = \|T(1)\|$ we obtain $T = 0$.

2. We have $\|T\| = r \iff T(\mathbb{1}) = r\mathbb{1}$ since $\|T\| = \|T(\mathbb{1})\| = r \iff T(\mathbb{1}) \leqslant r\mathbb{1} \iff T(\mathbb{1}) = \mathbb{1}$ since φ is faithful.

3. If $r \in P\sigma(T)$, then T is equivalent to a positive irreducible operator S with $S(\mathbb{1}) = \mathbb{1}$ (see the proof of Lemma 2.1.4).

4. There are positive and irreducible operators with empty point spectrum. An example on the commutative C*-algebra $C([0,1])$ is given by the operator

$$(Tf)(s) = sf(s) + \int_0^s f(t)\mathrm{d}t + \int_0^1 (1-t)^2 f(t)\mathrm{d}t$$

   for $0 \leqslant s \leqslant 1$. Then $\|T\| = 7/3$, $\sigma(T) = [0,1]$ and $P\sigma(T) = \varnothing$. For this see Schaefer [21], p. 312.

The following theorem is in complete analogy to the commutative situation, since every positive operator is already completely positive. For this we refer to Schaefer [21], Satz 3 or Schaefer [22], Appendix Theorem 3.3.

Theorem 2.2.3 *Let $\mathfrak{A}$ be a C*-algebra and suppose that T is an irreducible completely positive operator on $\mathfrak{A}$ satisfying $T(\mathbb{1}) = \mathbb{1}$. Then the following assertions hold.*

1. *The fixed-space of T is one-dimensional.*

2. *The peripheral point spectrum of T is a subgroup of the circle group.*

3. *Each peripheral eigenvalue α of T is simple, the corresponding eigenfunction is unitary and $\sigma(T) = \alpha \cdot \sigma(T)$.*
4. *1 is the unique eigenvalue of T with a positive eigenvector.*
5. *If the peripheral point spectrum contains a pole of the resolvent, then all all its points are poles of order 1 of the resolvent.*

PROOF. For $x, y \in \mathfrak{A}$ define

$$B(x,y) := T(xy^*) - T(x)T(y)^*.$$

Then B is a sesquilinear, positive mapping from $\mathfrak{A}\times\mathfrak{A} \to \mathfrak{A}$ satisfying $B(x,x) = 0$ for some $x \in \mathfrak{A}$ iff $B(x,y) = 0$ for all $y \in \mathfrak{A}$.

By Prop. 2.2.1 there exists a state φ such that $T^*\varphi = \varphi$ and φ is faithful since T is irreducible.

1. If $x \in \mathrm{Fix}(T) = \{x \in \mathfrak{A} \colon Tx = x\}$, then $B(x,x) = 0$, hence $B(x,y) = 0$ or $T(xy) = xT(y)$ and $T(yx) = T(y)x$ for all $y \in \mathfrak{A}$. Hence $T(xy^*) = xy^*$ for all $x, y \in \mathrm{Fix}(T)$ and $\mathrm{Fix}(T)$ is a C*-subalgebra of $\mathfrak{A}$ with $\mathbb{1} \in \mathrm{Fix}(T)$.

Next choose $0 \leqslant x \in \mathrm{Fix}(T), \|x\| = 1$. Then $0 \notin \sigma(x)$: For if $0 \in \sigma(x)$, then there is a state ψ on $\mathfrak{A}$ such that $\psi(x) = 0$. But the face $\mathfrak{F}_x$ is T-invariant, hence dense in $\mathfrak{A}_+$, hence $\psi = 0$ on $\mathfrak{A}$.

If $\mathbb{1} \neq x$, then $0 \neq y := \mathbb{1} - x \in \mathrm{Fix}(T)$ and $0 \notin \sigma(y)$. But then

$$\mathbb{1} \leqslant ny = n\mathbb{1} - nx \quad \text{or} \quad x \leqslant \left(\frac{n-1}{n}\right)\mathbb{1} < \mathbb{1},$$

i.e. $\|x\| < 1$, a contradiction. Thus $\mathrm{Fix}(T) = [\mathbb{1}]$.

2 & 3. Let $\alpha \in P\sigma_\pi(T)$ with normalized eigenvector x_α. Then

$$T(x_\alpha x_\alpha^*) \geqslant T(x_\alpha)T(x_\alpha^*) = x_\alpha x_\alpha^*,$$

hence $x_\alpha x_\alpha^* \in \mathrm{Fix}(T)$ or $x_\alpha x_\alpha^* = \mathbb{1}$. Similarly one proves $x_\alpha^* x_\alpha = \mathbb{1}$, i.e. x_α is unitary.

From $B(x_\alpha, x_\alpha) = 0$ we obtain

$$B(x_\alpha, y) = 0 \quad \text{or} \quad \alpha x_\alpha T(y)$$

for all $y \in \mathfrak{A}$.

This implies $\alpha\beta^* \in \sigma(T)$ for all $\alpha, \beta \in P\sigma_\pi(T)$ since the product of unitary elements is always $\neq 0$.

From this we conclude that

$$\alpha T = M_\alpha \cdot T \cdot M_\alpha^{-1}$$

where M_α is the isometry $(y \mapsto x_\alpha^* y)$ on $\mathfrak{A}$. Since

$$\sigma(T) = \sigma(M_\alpha \cdot T \cdot M_\alpha^{-1}) = \alpha\sigma(T),$$

the assertion follows.

4. If $T(x) = \alpha x$ for some $0 \leqslant x$, then $\varphi(x) = \varphi(Tx) = \alpha\varphi(x)$. Thus $\alpha = 1$ because $\varphi(x) \neq 0$. ■

Remark 2.2.4. From the proof above we get the following: If there exists a primitive h-th root of unity in the peripheral spectrum of T, then there exists mutually orthogonal projections $p_0, \dots, p_{h-1}$ with sum $\mathbb{1}$ and $Tp_l = p_{l-1}$, $l \in \mathbb{Z}/(h)$.

In particular, if $\mathfrak{A}$ is a simple C*-algebra, then the peripheral point spectrum cannot contain points α such that α is a root of unity distinct from 1.

Remark 2.2.5. We will show later that under the assumptions of Theorem 2.2.3 (5) the peripheral spectrum of T consist entirely of eigenvalues which are necessarily poles of the resolvent (see Corollary 3.1.3).

For irreducible completely positive operators we obtain not only information about the peripheral point spectrum but also on the action of T on the subspace generated by the eigenvectors pertaining to the peripheral eigenvalues.

Theorem 2.2.6 *Let $\mathfrak{A}$ be a C^*-algebra and suppose that T is an irreducible completely positive operator on $\mathfrak{A}$ satisfying $T(\mathbb{1}) = \mathbb{1}$. If $\mathfrak{M}$ is the closed subspace of $\mathfrak{A}$ generated by the eigenvectors pertaining to the peripheral eigenvalues, then*

1. *$\mathfrak{M}$ is a C^*-subalgebra of $\mathfrak{A}$ with $\mathbb{1} \in \mathfrak{M}$.*
2. *The restriction of φ to $\mathfrak{M}$ is a faithful trace.*

3. The restriction of T to $\mathfrak{M}$ is a *-automorhism.

PROOF. Using the methods above we see that $\mathfrak{M}$ is a C*-subalgebra of $\mathfrak{A}$ and the restriction of T to $\mathfrak{M}$ is a *-automorhism.

If x_α, x_β are unitary eigenvectors with peripheral eigenvalues α, β, then

$$\varphi(x_\alpha x_\beta^*) = \varphi(T(x_\alpha x_\beta^*)) = \alpha\beta^*\varphi(x_\alpha x_\beta^*).$$

From this follows that φ acts a trace on $\mathfrak{M}$. ∎

Remark 2.2.7. Take $\alpha, \beta \in P\sigma_\pi(T)$ with unitary eigenvectors $u_\alpha, u_\beta \in \mathfrak{A}$. Then

$$T(u_\alpha u_\beta) = c(\alpha, \beta)u_{\alpha\beta}$$

with $c(\alpha, \beta) \in \mathbb{T}$. The map

$$c : P\sigma_\pi(T) \to \mathbb{T}$$

has the following properties:

$$\begin{aligned} c(\alpha, \beta)c(\alpha\beta, \gamma) &= c(\alpha, \beta\gamma)c(\beta, \gamma) \\ c(\alpha, 1) &= c(1, \alpha) = 1 \\ c(\beta^*, \alpha^*) &= c(\alpha, \beta)^*. \end{aligned}$$

The function c is called a *multiplier* and "measures" how non-commutative the algebra $\mathfrak{M}$ is. We will investigate in this later.

2.3 Two Examples

Example 1: Let $\mathfrak{H}$ be the Hilbert space $\ell^2(\mathbb{Z})$. For $n, m \in \mathbb{Z}$ we define the unitary operators

$$\begin{aligned} (U(n)\xi)(k) &= \xi(k-n), \\ (V(m)\xi)(k) &= \exp(-\mathrm{i}km)(\xi(k). \end{aligned}$$

Then for all $n, m \in \mathbb{Z}$

$$U(n)V(m) = \exp(\mathrm{i}nm)V(m)U(n). \tag{2.6}$$

Let $\mathfrak{M}$ be the closed C*-algebra generated by $\{U(n)V(m) \colon n, m \in \mathbb{Z}\}$ in $\mathcal{B}(\mathfrak{H})$.

Choose $t \in \mathbb{R}$ such that $1, t$ and 2π are linearly independent over $\mathbb{Z}$ and let $V(t)$ be the unitary operator

$$(V(t)\xi)(k) = \exp(\mathrm{i}tk)(\xi(k).$$

Then $U := U(1)V(t)$ is unitary in $\mathcal{B}(\mathfrak{H})$, the inner *-automorphismus T

$$x \mapsto UxU^*,\ x \in \mathcal{B}(\mathfrak{H}),$$

leaves $\mathfrak{M}$ invariant, is irreducible and completely positive and has peripheral point spectrum

$$P\sigma_\pi(T) = \{\exp(\mathrm{i}(nt - m)\colon n, m \in \mathbb{Z}.$$

Example 2: Consider the left regular representation λ of a discrete group G on $\ell^2(\mathrm{G})$. Let $\delta_t, t \in \mathrm{G}$ denote the canonical unit vectors in $\ell^2(\mathrm{G})$ and $\mathfrak{A}$ be the C*-algebra given by the norm closure of the linear span of $\{\lambda(s)\colon s \in \mathrm{G}\}$.

If h is a positive definite function on G with $h(e) = 1$ and $0 \neq h(s) \neq 1$ for all $e \neq s \in \mathrm{G}$, then the extension of

$$T_h(\lambda(s)) := h(s)\lambda(s)$$

to $\mathfrak{A}$ defines an irreducible, completely positive operator on $\mathfrak{A}$ with $T_h(\mathbb{1}) = \mathbb{1}$.

For a concrete example we consider $\mathrm{G}_1 := \mathbb{Z}$, take $\hat{x}$ a character on this group with $\hat{x}(n) \neq 1$ for all $0 \neq n \in \mathbb{Z}$ and take G_2 another discrete group. Let now χ be the characteristic function of the set $\{e\}$, e the unit of G_2 and for $s \in \mathrm{G}_2$ we set

$$f(s) := a(1 - \chi(s)) + \chi(s)$$

where $a \in \mathbb{R}$. Then f is positive definite on G_2.

The group

$$\mathrm{G} := \mathrm{G}_1 \times \mathrm{G}_2$$

is discrete and the function

$$h := ((n, s) \mapsto \hat{x}(n)f(s)\ ), \quad n, s \in \mathrm{G},$$

has the desired properties. Especially, the peripheral spectrum of T_h is given by the set $\{\hat{x}(n)\colon n \in \mathbb{Z}\}$ with eigenvectors $\{\lambda(n, e)\colon n \in \mathbb{Z}\}$.

3 The Peripheral Spectrum of Completely Positive Operators

3.1 Quasi Compactness

In section 2.2 we studied the peripheral point spectrum of an irreducible completely positive operator. In the following we study the entire peripheral spectrum. The main result is the following.

Theorem 3.1.1 *Let T be a completely positive operator on a C^*-algebra $\mathfrak{A}$. If 1 is a pole of the resolvent of T and if the fixed-space of T is finite-dimensional, then the peripheral spectrum of T consists entirely of poles of the resolvent of T, i.e. eigenvalues of T.*

Remark 3.1.2. If the fixed space of T is not finite-dimensional, then there may exist elements in the peripheral spectrum of T which are not poles of the resolvent $R(\mu, T)$. For an example let T_n be the positive operator on the (commutative C*-algebra $\mathbb{C}^2$ given by the matrix

$$\begin{pmatrix} 0 & \\ \mathbb{1} - n^{-1} & n^{-1} \end{pmatrix}, \quad n \in \mathbb{N}.$$

If $\mathfrak{A}$ is the ℓ^∞-product of $\mathbb{C}^2$ and $T = (T_n)$ on $\mathfrak{A}$, then T is a completely positive operator with 1 as pole of the resolvent,

$$\sigma(T) = \{1\} \cup \{-1\} \cup \{-1 + n^{-1} : n \in \mathbb{N}\}.$$

Thus -1 is not isolated in $\sigma(T)$, hence not a pole of the resolvent.

Corollary 3.1.3 *Suppose T is an irreducible completely positive operator on $\mathfrak{A}$ and suppose the spectral radius $r(T)$ is a pole of the resolvent. Then the peripheral spectrum of T is the group Γ_k of all k-th roots of* *unity consisting entirely of first order poles of the resolvent of T.*

Remark 3.1.4. In the situation of the corollary it is easy to see, that every peripheral eigenvalue is a pole of the resolvent. So one has to show that the peripheral spectrum of T belongs to the point spectrum. To do this, one has to develop ultrapower technology for non-commutative C*-algebras.

The complete proof has been published in Groh [10] but will be given later.

Remark 3.1.5. This result extends a famous result of Niiro and Swashima to the non-commutative setting.

The important point of Theorem 3.1.1 is the remarkable consequence for the convergence behavior of the powers of T and the Césaro means of T in the *uniform operator topology*. We will dig into this in Chapter 5.

Theorem 3.1.6 *If T is completely positive and identity preserving on $\mathfrak{A}$ and if 1 is an isolated point of the spectrum of T, then the peripheral spectrum of T is finite.*

If $\mathfrak{A}$ is commutative, this follows from the cyclicity of the peripheral spectrum of T (see e.g. Schaefer [23], V. Thm. 4.4). For the non-commutative case we will prove this later.

We close this section by a question:

Question 1: Let T be completely positive and identity preserving on $\mathfrak{A}$ and let 1 be a pole of the resolvent. If $\alpha \in \sigma_\pi(T)$ is isolated in $\sigma(T)$, does it follow that α is a pole of the resolvent of T ? In the commutative case this has been proven in Niiro & Sawashima [18], Thm, 7.

4 Spectral Theory on W*-Algebras

4.1 Basic Definitions and Techniques

In the following section we look at operators on W*-algebras. Since every W*-algebra is also a C*-algebra, nothing new comes up if we consider operators which are just norm continuous.

However, if we consider completely positive operators with respect to the duality $\mathfrak{A}$ and its predual $\mathfrak{A}_*$, i.e. operators which are weak*-continuous on $\mathfrak{A}$, new aspects appear. If T is such an operator, it possesses a preadjoint T_* which is positive if T is. We call T_* *completely positive*, iff T has this property.

Since the norm closed faces of $\mathfrak{A}_*$ and the weak*-closed faces of $\mathfrak{A}$ are in duality, the definition of *irreducibility* is now the following.

Definition 4.1.1. Let T be a completely positive operator on $\mathfrak{A}_*$. Then T_* is called *irreducible* if no closed non-trivial face of $\mathfrak{A}_+^*$ is invariant under T_*.

By duality, this is equivalent to: No non-trivial weak*-closed face of $\mathfrak{A}_+$ is invariant under T.

We need some techniques , which we summarize here. First we recall some facts about weak* continuous linear forms, that is elements of the predual of $\mathfrak{A}_*$. If $\varphi \in \mathfrak{A}_*$, then there exist a positive linear form $|\varphi| \in \mathfrak{A}_*$ and a partial isometry $u \in \mathfrak{A}$ such that

$$\varphi(x) = |\varphi|(xu) =: (R_u|\varphi|)(x),\; x \in \mathfrak{A}$$
$$u^*u = s(|\varphi|),$$

where $s(|\varphi|)$ is the *support projection* of $|\varphi|$. We refer to this as the *polar decomposition* of $|\varphi|$. In addition, $|\varphi|$ is uniquely determined by the properties

$$\|\varphi\| = \||\varphi|\| = \||\varphi|(\mathbb{1})\|$$
$$|\varphi(x)|^2 \leqslant \|\varphi\||\varphi|(xx^*).$$

From this we obtain the following useful lemma.

Lemma 4.1.2 *Let $\mathfrak{A}$ be a W*-algebra with predual $\mathfrak{A}_*$ and let T be completely positive and identity preserving on $\mathfrak{A}$ with preadjoint T_*. If α is a peripheral eigenvalue of T_* with normalized eigenvector φ_α, then $|\varphi_\alpha|$ and $|\varphi_\alpha^*|$ are elements of the fixed space Fix(T_*).*

PROOF. Using the Schwarz inequality for T, we obtain

$$|\varphi_\alpha(x)|^2 = |\varphi_\alpha(Tx)|^2 \leqslant |\varphi_\alpha|((Tx)(Tx)^* \leqslant (T_*|\varphi_\alpha(x)|(xx^*)$$

for $x \in \mathfrak{A}$. Because $\|\varphi_\alpha\| = \|T_*\||\varphi_\alpha|$, the assertion follows because of the characterizing properties of the polar decomposition. ■

Remark 4.1.3. If T is completely positive and identity preserving on $\mathfrak{A}$ with preadjoint T_* and if the peripheral point spectrum of the preadjoint operator is not empty, then there is always a positive normal linear form φ in the fixed space of T_*. In particular, if T_* is irreducible, then such a φ is faithful on $\mathfrak{A}$.

Next we mention a result from the spectral theory of operators on Banach spaces.

Lemma 4.1.4 *Let T be a contraction on a Banach space X. Then for every peripheral eigenvalue α of T, ker$(\alpha - T^*)$ separates the points of ker$(\alpha - T)$. In particular, $P\sigma_\pi(T) \subseteq P\sigma_\pi(T^*)$ and dim ker$(\alpha - T) \leqslant$ dim ker$(\alpha - T^*)$.*

PROOF. Since $\ker(\alpha - T^*) = \mathrm{Fix}(\alpha^* T^*)$ it suffices to prove this for the fixed spaces of T and T_*. This can be done following the proof of Prop. 2.2.1 and we omit the details. ■

4.2 Spectrum on the Predual

For the deeper investigations we need the following lemma. Recall that we call the triple $(\mathfrak{A}, \varphi, T)$ a *W**-*dynamical system*, if

- $\mathfrak{A}$ is a W*-algebra,
- φ is a faithful normal state on $\mathfrak{A}$
- T is completely positive and identity preserving with $T^*\varphi = \varphi$.

Then T has a preadjount T_* on $\mathfrak{A}_*$.

We call $(\mathfrak{A}, \varphi, T)$ an *irreducible W**-*dynamical system*, if T_* is irreducible on $\mathfrak{A}_*$.

Lemma 4.2.1 *Let $(\mathfrak{A}, \varphi, T)$ be a W*-dynamical system. Then the cyclic semi-group*

$$\mathfrak{S}_* := \{T_*^n : n \in \mathbb{N}\}$$

is relatively compact in the weak operator topology $\mathcal{L}_w(\mathfrak{A}_*)$.

Since this situation is discussed in greater detail in a forthcoming paper (see [12]), we omit the details just mentioning that it follows from the characterization of weakly-compact sets in the predual of a W*-algebra (see e.g. Takesaki [25], III Thm. 5.4).

Proposition 4.2.2 *Let $(\mathfrak{A}, \varphi, T)$ be an irreducible W*-dynamical system. Then the following assertions hold.*

1. *The set of peripheral eigenvalues of T and T_* are equal.*
2. *There exist a faithful, normal conditional expectation Q of $\mathfrak{A}$ onto the weak*-closed linear span $\mathfrak{M}$ of all eigenvectors of T pertaining to the peripheral eigenvalues.*

The proof of (1) uses Lemma 4.1.4 and the techniques of Theorem 2.1.7. The second part is a consequence of the theory of compact semigroups and will be worked out in Groh, Kunszenti-Kovacs, Schreiber & Batkai [12].

Proposition 4.2.3 *Let $(\mathfrak{A}, \varphi, T)$ be an irreducible W^*-dynamical system. Then the following assertions hold.*

1. *The peripheral spectra of T and T_* are equal.*
2. *The peripheral spectrum of T is a subgroup of the circle group.*
3. *Every peripheral eigenvalue α of T or T_* is simple.*
4. *$\sigma(T) = \alpha\sigma(T)$ and $\sigma(T_*) = \alpha\sigma(T_*)$.*

Remark 4.2.4. Here we will state some more properties of eigenvectors of irreducible T_*.

4.3 Example: Dynamical Systems on $\mathcal{B}(\mathfrak{H})$

In this section we restrict a little further an we are looking for irreducible W^*-dynamical systems on $\mathcal{B}(\mathfrak{H})$.

Theorem 4.3.1 *Let $(\mathfrak{H}, \varphi, T)$ be an irreducible, W^*-dynamical system. Then the peripheral spectrum of T is the group of all h-th roots of unity for some $h \geqslant 1$.*

PROOF. This is part of [9] ■

5 The Convergence Properties of Completely Positive Operators

5.1 Convergence Properties on W*-Algebras

If T is a contraction on a finite-dimensional Banach space, then the peripheral eigenvalues determines the asymptotic behavior of the powers T^n of T. In order to extend these results to infinite-dimensional Banach spaces, the first difficulty is of topological nature, i.e., one has to distinguish between the various operator topologies.

The second difficulty is of spectral theoretical nature. The spectral values of T need not be eigenvalues nor are the singularities of the resolvent of T necessarily poles.

In this section we show how to solve these problems if we consider completely positive operators on W*-algebras.

More precisely, we investigate the following problem:

Problem: *Let $\mathfrak{A}$ to be a W*-algebra with predual $\mathfrak{A}_*$ and T to be completely positive with pre-adjoint T_*. Under what conditions on the spectrum of T in a neighborhood of 1 and in which operator topology does the sequence T^n, $n \in N$ converge.*

Before dealing with the powers of T we recall the results on the convergence of the *Césaro means*

$$T_n := \sum_{k=0}^{n-1} T^k \quad \text{resp.} \quad T_n^* := \sum_{k=0}^{n-1} T_*^k.$$

If X is a Banach space and $T\mathcal{L}(\mathfrak{X})$, then we call T *mean ergodic*, if the Césaro means converges in the strong operator topology and we call T *uniformly* *ergodic*, if the Césaro means converges in the operator topology. The limit is in both cases a projection P onto the fixed space of T.

For example, if 1 is pole of the resolvent of T, then T, then T is uniformly ergodic.

Theorem 5.1.1 *Let* $(\mathfrak{A}, \varphi, T)$ *a* W*-dynamical system. *The the following assertions hold.*

1. *The Césaro means of* T_* *(resp.* T *) converge in the strong operator topology* $\mathcal{L}_{\mathrm{s}}(\mathfrak{A}_*)$ *(resp. weak*-operator topology* $\mathcal{L}_{\mathrm{w}*}(\mathfrak{A})$ *to a projection* P_* *(resp.* P*) onto the fixed space of* T_* *(resp.* T*).*
2. *The Césaro means of* T_* *(resp.* T *) converge in the operator topology if* 1 *is a pole of the resolvent of* T_**, hence of* T*.*

The assertion 5.1.1 (1) follows e.q. from Schaefer [23], III. Example 7.3. whereas 5.1.1 (2) is a well known fact from general spectral theory in Banach spaces (see e.g. Eisner [5]).

Definition 5.1.2. We call an operator S on a Banach space X *partially periodic* if there exists $m_0 \in \mathbb{N}$, such that

$$S(I - S^{m_0}) = 0.$$

Letting

$$X_0 := (I - S^{m_0})(X) \quad \text{and} \quad X_1 := S^{m_0}(X)$$

it follows that

$$X = X_0 \oplus X_1 \quad \text{and} \quad S_{|X_0} = \mathrm{O} \quad \text{and} \quad \left(S_{|X_1}\right)^{m_0} = I_{|X_1}.$$

In particular, every periodic operator is partially periodic with $X_0 = \{0\}$.

Theorem 5.1.3 *(Uniform Convergence) Let* $(\mathfrak{A}, \varphi, T)$ *a W*-dynamical system. The the following are equivalent:*

1. 1 *is a pole of the resolvent.*
2. *There exists a partially periodic W*-dynamical system* $(\mathfrak{A}, \varphi, S)$

such that $\lim_n(S^n - T^n) = 0$ *in the norm operator topology* $\mathcal{L}(\mathfrak{A})$.

PROOF. (1) $\Longrightarrow$ (2): The assumption implies that T is quasi-compact, hence uniformly ergodic and $\sigma_\pi(T)$ is the group Γ_h of all h$^{\text{th}}$-roots of unities for some $h \in \mathbb{N}$. Therefore, the operator T^h is uniformly ergodic mean with $\sigma_\pi(T^h) = \{1\}$ and the projection Q of $\mathfrak{A}$ onto the fixed space of T^h is completely positive, preserves the identity, and is faithful (i.e. $\ker(T) \cap \mathfrak{A}_+ = \{0\}$).

Letting

$$S := T \circ Q,$$

then S is irreducible, is partially periodic because of $S^h = Q$ and

$$T^n - S^n = (T(I - Q)) \to O$$

in the norm operator topology since $r(T(I - Q)) < 1$.

(2) $\Longrightarrow$ (1): S is partially periodic, thus S^{m_0} is a projection and $T^{k \cdot m_0} \to S^{m_0}$ in the norm operator topology. Hence T^{m_0} is uniformly ergodic whence 1 is a pole of the resolvent of T^{m_0} and thus of T. ■

Since for a W*-dynamical system $(\mathfrak{A}, \varphi, S)$ the operator T_* is mean ergodic in $\mathcal{L}_s(\mathfrak{A}_*)$, there is a splitting of $\mathfrak{A}_*$ into T_*-invariant subspaces

$$\mathfrak{A}_* = \text{Fix}(T_* \oplus \ker P_*, \tag{5.1}$$

where P_* is the associated mean ergodic projection.

Since the cyclic semi-group $\mathfrak{S}_* := \{T_*^n : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{L}_w(\mathfrak{A}_*)$, there is another splitting of $\mathfrak{A}_*$ in T_*-invariant subspaces

$$\mathfrak{A}_* = Q_*(\mathfrak{A}_*) \oplus \ker Q_*, \tag{5.2}$$

where Q_* is the unique projection in the closure of $\mathfrak{S}_*$ in $\mathcal{L}_w(\mathfrak{A}_*)$ and the range of Q_* is the closed linear subspace in $\mathfrak{A}_*$ generated by the peripheral eigenvalues.

Following Nagel [16], we call a W*-dynamical system

- *extremely ergodic*, if $P_* = Q_*$
- *mixing*, if T_* is irreducible and extremely ergodic.

The proof of the following theorem follows from Nagel [16], Thm. 2.6 and the fact that $P\sigma(T) \cap \Gamma = P\sigma(T_*) \cap \Gamma$.

Theorem 5.1.4 *Let $(\mathfrak{A}, \varphi, T)$ a W^*-dynamical system and suppose $\mathfrak{A}$ to be separabel. Then the following assertions are equivalent.*

1. *$(\mathfrak{A}, \varphi, T)$ is mixing.*
2. *$(\mathfrak{A}, \varphi, T)$ is irreducible and 1 is the unique peripheral eigenvalue of T.*
3. *There exists a sequence $(k(n))_{n\in\mathbb{N}}$ in $\mathbb{N}$, such that*

$$\lim_n (T_*^{k(n)} - \mathbb{1} \otimes \varphi) = 0$$

in the weak operator topology $\mathcal{L}_{\mathrm{w}}(\mathfrak{A}_*)$.

A sequence $(k(n))_{n\in\mathbb{N}}$ in $\mathbb{N}$ has density $\alpha \in [0,1]$ if

$$\lim_m \frac{1}{m} |\{k(n) \colon n \in \mathbb{N}\} \cap \{1, \dots, m\}| = \alpha$$

For more informations about this see e.g. Eisner [5].

Theorem 5.1.5 *(Weak Convergence) Let $(\mathfrak{A}, \varphi, T)$ an irreducible W^*-dynamical system and suppose $\mathfrak{A}$ to be separabel. Then the following assertions are equivalent.*

1. *1 is isolated in the peripheral point spectrum of T.*
2. *There exists a sequence $(k(n))_{n\in\mathbb{N}}$ in $\mathbb{N}$, such that*

$$\lim_n (T_*^{k(n)} - \mathbb{1} \otimes \varphi) = 0$$

in the weak operator topology $\mathcal{L}_{\mathrm{w}}(\mathfrak{A}_*)$.

Proof. Under the assumptions, the peripheral point spectrum of T is the group of all h-th roots of unity for some $h \geqslant 1$. By the following lemma therefore we have $P\sigma(T^h) = \{1\}$. Using Thm. 5.1.4 the assertion follows as in the proof of Thm 5.1.4. ∎

Lemma 5.1.6 *Let T be a contraction on a Banach space X. Then $(P\sigma(T) \cap \Gamma)^n = P\sigma(T^n) \cap \Gamma$ for all $n \in \mathbb{N}$.*

Proof. Obviously $(P\sigma(T) \cap \Gamma)^n \subseteq P\sigma(T^n) \cap \Gamma$. Conversely, take $\alpha \in P\sigma(T^n) \cap$

Γ and $\gamma \in \Gamma$ such that $\gamma^n = \alpha$. For $0 \leqslant k \leqslant n-1$ and ϵ a primitive n-th root unity consider

$$y_k := \frac{1}{n} \sum_{s=0}^{n-1} \gamma^s \epsilon^{-sk} T^s x.$$

Since $\sum_k = x$ and $T y_k = \gamma \epsilon^k y_k$ the assertion follows. ■

Example 5.1.7. Every irreducible irreducible W*-dynamical system on $\mathcal{B}(\mathfrak{H})$ fulfills assertion (1) of Thm. 5.1.4.

Theorem 5.1.8 *(Strong Convergence) Let $(\mathfrak{A}, \varphi, T)$ an irreducible W^*-dynamical system. If 1 is isolated in the peripheral spectrum of T, then there exists an irreducible, partially periodic system W^*-dynamical system $(\mathfrak{A}, \varphi, S)$ such that*

$$\lim_n (T_*^n - S_*^n) = 0$$

in the strong operator topology $\mathcal{L}_s(\mathfrak{A}_*)$

We only sketch the proof which will be published in details separately.

1. It follows from Thm. 3.1.6, that the peripheral spectrum is finite.
2. Since T is bibounded in the sense of Groh & Kümmerer [11], there is an extension of T to an operator T_π on the standard Hilbert space $(\mathfrak{H}, \mathcal{P}, J)$ associated with $(\mathfrak{A}, \varphi)$. Using the topological properties of the canonical injections j_1 and j_2 introduced in [11], Sec. 2, it follows that the peripheral spectrum of T_φ is finite, in particular has Lebesgue measure zero.
3. Since $\|T_\varphi\| \leqslant 1$, we find a splitting of $\mathfrak{H}$ into T_φ invariant subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$, such that $T_{|\mathfrak{H}_0}$ is unitary and $T_{|\mathfrak{H}_1}$ is completely non-unitary. Using Sz.-Nagy & Foias [24], II. Prop. 6.7 and the group property of the peripheral point spectrum of T, it turns out that there exists a partially periodic operator S_φ on $\mathfrak{H}$ such that

   $$\|T_\varphi^n \xi - S_\varphi^n \xi\| \to 0$$

   for all $\xi \in \mathfrak{H}$.
4. Using [11] again and keeping in mind that $S_\varphi(\mathcal{P}) \subseteq \mathcal{P}$, the operator S_φ

defines via the injection $j_2 \colon \mathfrak{H} \to \mathfrak{A}_*$ an irreducible, partially periodic W*-dynamical system $(\mathfrak{A}, \varphi, S)$ on $\mathfrak{A}_*$ with the desired properties.

Remark 5.1.9. If $(\mathcal{B}(\mathfrak{H}), \varphi, T)$ is an irreducible W*-dynamical system, then there always exists an irreducible, partially periodic system W*-dynamical system $(\mathcal{B}(\mathfrak{H}), \varphi, S)$ such that $\lim_n (T_*^n - S_*^n) = 0$ in $\mathcal{L}_s(\mathcal{B}(\mathfrak{H})_*)$

5.2 Convergence in the Strong Operator Topology of $\mathcal{L}(\mathfrak{A})$

In the last sections we investigated into the convergence properties of T^n for the various operator topology on $\mathfrak{A}_*$. It makes no sense to investigate also the strong operator topology on $\mathfrak{A}$ by in the following fact.

Theorem 5.2.1 *Let $\mathfrak{A}$ be a W*-algebra and let $0 \leqslant T \in \mathcal{L}(\mathfrak{A})$. If T^n converges in the strong operator topology of $\mathcal{L}(\mathfrak{A})$, then T^n converges in the norm operator topology.*

For the proof see Groh [8].

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